

# Lorentz Covariant Quantum 4-Potential and Orbital Angular Momentum for the Transverse Confinement of Matter Waves

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In two recent papers exact Hermite-Gaussian solutions to relativistic wave equations have been obtained for both electromagnetic and particle beams that include Gouy phase. The solutions for particle beams correspond to those of the Schrödinger equation in the non-relativistic limit. Here, distinct canonical and kinetic 4-momentum operators will be defined for quantum particles in matter wave beams. The kinetic momentum is equal to the canonical momentum minus the fluctuating terms resulting from the transverse localization of the beam. Three results are obtained. First, the total energy of a particle for each beam mode is calculated. Second, the localization terms couple into the canonical 4-momentum of the beam particles as a Lorentz covariant quantum 4-potential originating at the waist. The quantum 4-potential plays an analogous role in relativistic Hamiltonian quantum mechanics to the Bohm potential in the non-relativistic quantum Hamilton-Jacobi equation. Third, the orbital angular momentum (OAM) operator must be defined in terms of canonical momentum operators. It is further shown that kinetic 4-momentum does not contribute to OAM indicating that OAM can therefore be regarded as a pure manifestation of quantum 4-potential.

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## I. INTRODUCTION

Experiment shows that beams of particles still behave like beams even if only one particle is traveling through the apparatus at a time [1]. The converse of this argument is that isolated particles can behave like beams. Specifically, it is understood that the wavefunction  $\Psi$  for the particle must take account of a full compliment of wave beam features such as mode numbers [2], Gouy phase [3, 4] and orbital angular momentum [5].

The purpose of this paper is to explain the localizing affect of transverse confinement on a beam particle using a quantum 4-potential. The concept of a quantum 4-potential as it is introduced here is similar to the Bohm potential [6] in the the sense that it is a concept extracted from  $\Psi$  rather than a representation of a field separate from  $\Psi$ . The point of departure is that the quantum 4-potential couples into each individual component of the 4-momentum operators for the particle whereas the scalar Bohm potential is a component in a Hamilton-Jacobi equation belonging to an alternate formulation of quantum mechanics. The quantum 4-potential therefore requires a distinction to be made between the canonical  $\hat{p}_\mu$  ( $\mu = 1, 2, 3, 4$ ) and kinetic  $\hat{P}_\mu$  4-momentum for the particle. The canonical (total) 4-momentum is the sum of the kinetic 4-momentum and the quantum 4-potential term.

Free particles have no quantum potential but localized particles do have it. The signature of a quantum potential is therefore the appearance a term in a quantum mechanical equation that generates localization and has no association to external source. It will be shown that

the form of this term depends on the specific formulation of quantum mechanics under consideration but that all the variants interrelate and have two distinct common properties; they vanish in the free particle limit and have null expectation values.

External devices are responsible for collimating and focusing the particles in a beam. Once a particle has passed through these devices, it remains localized but is no longer confined. Our solutions describe the localized state of the particles but not the passage of the particle through the devices responsible for confining the beam.

The basic structure of a wave beam can be understood using the Heisenberg uncertainty principle [7] that states uncertainty in momentum is inversely proportional to uncertainty in position. In a continuous wave beam there is no localization of the particle along the axis of the beam meaning that each particle can be assumed to have a precise axial momentum and therefore a precise axial velocity  $v_3$ . The uncertainty in the position of the particle along the transverse axis is smallest at the beam waist. It is therefore the size of the waist that determines the uncertainty in the transverse momentum of the particle. The presence of transverse momentum explains the fact beams spread. It also accounts for the existence of orbital angular momentum in beams.

Linear wave equations have both plane-wave and localized solutions [8] often called wave packets [9]. The wave packet is smallest at the time of an event that localizes the particle then continuously grows in size afterwards. One distinguishing characteristic of plane-wave and localized wave functions is the number of 4-position dependencies in them. Plane-waves are local functions that only depend on the position  $x_i$  ( $i = 1, 2, 3$ ) of the

particle at time  $t$ . By contrast, localized wave solutions are bilocal functions since they must depend on both the current 4-position of the particle as well as the 4-position  $X_\mu = (X_i, cT)$  of the preceding confinement event where and when the size of the wave packet was at a minimum. It is the bilocal nature of wave packets that permits the probability density of finding free particles to have spatial extension as well as a 4-position. It is also the dependence of  $\Psi$  on  $X_\mu$  as well as  $x_\mu$  that will enable us to define distinct kinetic  $P_\mu$  and canonical  $p_\mu$  4-momentum vectors.

Bateman-Hillion functions [10, 11] are exact localized solutions of relativistic wave equations that trace back to early work of Bateman on conformal transformations [12]. In two recent papers, exact Bateman-Hillion solutions were obtained for the Hermite-Gaussian modes of both electromagnetic [13] and quantum particle [14] beams. These are detailed solutions for particle beams that include Gouy phase [15–17]. The paraxial wave equation [2] for electromagnetic beams and the Schrödinger equation for non-relativistic particle beams have both been demonstrated as limiting cases of the Bateman-Hillion method.

One method of obtaining Bateman-Hillion solutions to a wave equation is to start from an ansatz. In the case of the Klein-Gordon equation the ansatz eliminates the second order time derivative reducing the wave equation to a parabolic form. This resolves problems of negative energies and negative probability densities that afflict the unconstrained Klein-Gordon equation. It will be further shown in this paper that the probability density of finding a particle in a Bateman-Hillion beam is just  $|\Psi|^2$  similar to the Schrödinger equation except that the probability density for Bateman-Hillion solutions is also form preserving under Lorentz transformations.

In this paper a transformation will be made to the Bateman-Hillion solutions of the Klein-Gordon equation for particle beams to account for an earlier finding [14] that the components of the 4-momentum of the particles must have a shift in them related to the complex shift in the 4-position coordinates needed for the accurate description of any wave beam. This will be shown to facilitate a calculation for the total energy of each particle in terms of the rest mass of the particle, the kinetic energy of the propagation of the particle along the axis of the beam and the kinetic energy locked up in the transverse mass flows. Results will be presented for both Hermite-Gaussian and Laguerre-Gaussian beams. Laguerre-Gaussian beams are useful to describe the orbital angular momentum states of the particle.

After the seminal paper by Bliokh et al introducing vortex beams carrying OAM for free quantum electrons [18] several experimental [19] and theoretical [20, 21] results were obtained. Properties of the interaction of OAM with an electric field such as OAM Hall effect was studied in the non relativistic context [18]. Further the interac-

tion of OAM with a magnetic field was also studied in the non relativistic context [22]. More recently the effect of the interaction of relativistic electron vortex beam with a laser field was studied showing that the beam center is shifted and that the shift in the paraxial beams is larger than that in the nonparaxial beams [23, 24]. The results that we are obtaining in this paper could be useful to explore the relativistic effects in the properties such as OAM Hall and Zeeman effect resulting, respectively, of the interaction of relativistic scalar (without spin) electron vortex beam with an electric and magnetic field. Further we can similarly solve the Dirac equation to include the effects of the interaction of spin angular momentum (SAM) with a magnetic field.

It will be shown in this paper that the Schrödinger and Klein-Gordon equations give the same orbital angular momentum for each scalar mode of a Laguerre-Gaussian beam. To find relativistic corrections to orbital angular momentum it is therefore necessary to investigate solutions that mix multiple modes. For example, in the case of Bessel beam solutions to the Dirac equation it has been found [21] the corrective amplitude coefficients take the form  $a = \sqrt{1 - E_0/E} \sin \theta_0$  where  $E$  denotes the energy of each particle,  $E_0$  is the rest energy and  $\theta_0$  is the polar angle indicating the divergence of the beam. This results in a relativistic correction  $sa^2$  to the total angular momentum of each particle with spin  $s$ . The correction clearly vanishes in both the non-relativistic ( $E \rightarrow E_0$ ) and paraxial ( $\theta_0 \rightarrow 0$ ) limits but can otherwise affect the energies of beam particles in external electric and magnetic fields. Another source for relativistic corrections that may affect OAM is the repulsion between charged particles. This can be a stronger effect than the spin-orbit interaction that could be studied using either the Klein-Gordon or Dirac equations. The repulsion between charged particles is also known to have a greater affect on the beam for lower energy particles.

The fact  $\Psi(x_i, t, X_i, T)$  depends on two 4-position vectors requires the introduction of a constraint condition [25, 26] to eliminate one of the independent time coordinates in the calculation of the physical properties for the beam. As in an earlier paper [14] the solution to be applied here is to use Dirac delta function notation to impose a relationship  $\xi_3 - v_3\tau = 0$  between the relative position  $\xi_i = x_i - X_i$  and relative time  $\tau = t - T$ . This relates back to the idea that particles in continuous wave beams can be assigned a precise axial velocity  $v_3$ .

In Sec. II, we use the Bateman-Hillion ansatz to solve the Klein-Gordon equation for a particle that passes through a beam waist. In Sec. III, we determine the Lorentz invariant probability density of finding a particle in a Bateman-Hillion beam. In Sec. IV, we calculate the kinetic 4-momentum in terms of the canonical 4-momentum and the localization terms. In Sec. V, we calculate the quantum 4-potential. In Sec. VI, we conclude our results in a summary.

## II. BATEMAN-HILLION BEAMS

Consider a beam of particles each having a rest mass  $m_0$ , a 4-position  $x_\mu = (x_i, ct)$  and a 4-momentum  $p_\mu = (p_i, E/c)$ . Let us assume each particle passes through a beam waist with a position  $X_i$  at the time  $T$ . The Klein-Gordon equation for the wave function  $\Psi(x_i, t, X_i, T)$  representing each of the particles in Minkowski space can be expressed as

$$\hat{p}_\mu \hat{p}^\mu \Psi = \frac{1}{c^2} (\hat{E}^2 - c^2 \hat{p}_i^2) \Psi = m_0^2 c^2 \Psi, \quad (1)$$

where

$$\hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial x_i}, \quad \hat{E} = -\frac{\hbar}{i} \frac{\partial}{\partial t}, \quad (2)$$

are the canonical 4-momentum operators,  $\hbar$  is Planck's constant divided by  $2\pi$  and  $c$  is the velocity of light.

One approach to solving eq. (1) for a beam is to use a Bateman inspired ansatz. In an earlier paper [14], the following trial form was taken as the starting point for the derivation of the positive-energy Hermite-Gaussian beam solutions

$$\Psi_{mn}^O = \Phi_{mn}(\xi_1, \xi_2, \xi_3 + c\tau) \exp[i(k_3 x_3 - k_4 ct)], \quad (3)$$

where

$$\xi_i = x_i - X_i, \quad \tau = t - T, \quad (4)$$

gives the position of each point  $x_\mu$  relative to the 4-position of the beam waist,  $k_\mu = (0, 0, k_3, k_4)$  is the wave vector and  $\Phi_{mn}$  are scalar functions. The positive integers  $m$  and  $n$  indicate the mode of the beam.

A curious feature of eq. (3) derived in [14] is that it leads to the following expression for the particle current in a Gaussian beam

$$\langle \Psi_{00}^O | \hat{j}_\mu | \Psi_{00}^O \rangle = \frac{\hbar}{m_0} (k_\mu - \kappa_\mu^{00}), \quad (5)$$

where

$$\Psi^* \hat{j}_\mu \Psi = \frac{1}{2m_0} (\Psi^* \hat{p}_\mu \Psi - \Psi \hat{p}_\mu \Psi^*), \quad (6)$$

and  $\kappa_\mu^{mn} = (0, 0, \kappa^{mn}, -\kappa^{mn})$ . Here, the axial parameter  $\kappa^{mn}$  takes the form

$$\kappa^{00} = \frac{1}{(k_3 + k_4)w_0^2}, \quad (7)$$

where  $w_0$  is the radius of the beam at the waist.

Eq. (5) suggests that  $k_\mu$  is related to the expectation value of the axial current for a particle in a beam. In seeking an intuitive definition for  $k_\mu$  we shall now make use of the unitary transformation

$$\Psi_{mn} = \Psi_{mn}^O \exp[i\kappa^{mn}(x_3 + ct)], \quad (8)$$

where

$$\kappa^{mn} = \frac{N^{mn}}{(k_3 + k_4)w_0^2}, \quad (9)$$

and  $N^{mn}$  is a constant. The general form of  $N^{mn}$  is to be determined but it can be seen from comparison of eqs. (7) and (9) that  $N^{00} = 1$ . It is also readily verified that eq. (8) is form invariant under the Lorentz transformation equations:

$$x'_3 = (x_3 - v_3 \tau) \gamma, \quad \tau' = (\tau - \frac{v_3}{c^2} x_3) \gamma \quad (10)$$

$$k'_3 = (k_3 - \frac{v_3}{c} k_4) \gamma, \quad k'_4 = (k_4 - \frac{v_3}{c} k_3) \gamma \quad (11)$$

where  $\gamma = 1/\sqrt{1 - v_3^2/c^2}$ . Applying the transformation (8) to eq. (3) gives

$$\Psi_{mn} = \Phi_{mn}(\xi_1, \xi_2, \xi_3 + c\tau) \times \exp[i(k_3 + \kappa^{mn})x_3 - i(c k_4 - \kappa^{mn})t], \quad (12)$$

equivalent to making the replacements  $k_3 \rightarrow k_3 + \kappa^{mn}$  and  $k_4 \rightarrow k_4 - \kappa^{mn}$ . These replacements can be used, in turn to reduce eq. (5) to the simplified to the form

$$\langle \Psi_{00} | \hat{j}_\mu | \Psi_{00} \rangle = \frac{\hbar}{m_0} (0, 0, k_3, k_4), \quad (13)$$

where it can be seen  $\kappa^{00}$  has been eliminated. One important goal of this paper will be to show that there exists  $N^{mn}$  such that the condition

$$\langle \Psi_{mn} | \hat{j}_\mu | \Psi_{mn} \rangle = \frac{\hbar}{m_0} (0, 0, k_3, k_4), \quad (14)$$

is satisfied. If this hypothesis is true, it implies  $\frac{\hbar}{m_0} k_\mu$  can be interpreted as the expectation value for the particle current in a relativistic beam thus giving a clear physical meaning to  $k_\mu$ .

Inserting eq. (12) into the Klein-Gordon equation (1) gives

$$\frac{\partial^2 \Phi_{mn}}{\partial x_1^2} + \frac{\partial^2 \Phi_{mn}}{\partial x_2^2} + 2i(k_3 + \kappa^{mn}) \frac{\partial \Phi_{mn}}{\partial x_3} + \frac{2i}{c} (k_4 - \kappa^{mn}) \frac{\partial \Phi_{mn}}{\partial t} = 0, \quad (15)$$

where

$$k_4^2 = k_3^2 + 2\kappa^{mn}(k_3 + k_4) + \frac{m_0^2 c^2}{\hbar^2}. \quad (16)$$

It can be seen the unitary transformation (8) has introduced the term

$$K_T^{mn} = 2\kappa^{mn}(k_3 + k_4), \quad (17)$$

into this dispersion relationship. The physical interpretation of  $K_T^{mn}$  will be discussed later once the relativistic

energy formula for each particle in the beam has been derived.

It is instructive to observe that

$$\frac{\partial}{\partial x_3} \Phi_{mn} = \frac{1}{c} \frac{\partial}{\partial t} \Phi_{mn}, \quad (18)$$

and equivalently

$$\frac{\partial}{\partial x_3} |\Psi_{mn}|^2 = \frac{1}{c} \frac{\partial}{\partial t} |\Psi_{mn}|^2, \quad (19)$$

owing the fact  $\Phi_{mn}$  only depends on  $\xi_3$  and  $\tau$  in the linear combination  $\xi_3 + \tau$ . Eqs. (15) and (18) can now be combined to obtain the operator relationships

$$\hat{p}_3 \Phi_{mn} = -\hat{p}_4 \Phi_{mn} = -\frac{\hat{p}_1^2 + \hat{p}_2^2}{2\hbar(k_3 + k_4)} \Phi_{mn}, \quad (20)$$

These results will prove useful later.

Equation (15) can be solved analogously to the paraxial equation [2] to give

$$\Phi_{mn} = \frac{C_{mn}^{HG} w_0}{w} H_m \left( \frac{\sqrt{2}\xi_1}{w} \right) H_n \left( \frac{\sqrt{2}\xi_2}{w} \right) \times \exp \left[ \frac{i2b(\xi_1^2 + \xi_2^2)}{w_0^2(\xi_3 + c\tau - i2b)} - i g_{mn} \right], \quad (21)$$

where  $H_m$  and  $H_n$  are Hermite polynomials,

$$b = \frac{w_0^2}{4} (k_3 + k_4), \quad (22)$$

$$w(\xi_3, \tau) = w_0 \sqrt{1 + \left( \frac{\xi_3 + c\tau}{2b} \right)^2}, \quad (23)$$

is the beam radius such that  $w_0 = w(0, 0)$  and

$$g_{mn}(\xi_3, \tau) = (1 + m + n) \arctan \left( \frac{\xi_3 + c\tau}{2b} \right), \quad (24)$$

is the Gouy phase of a relativistic quantum particle.

It is notable that the Klein-Gordon equation (1) can also be usefully solved in cylindrical coordinates starting from the expression

$$\Psi_{lp} = \Phi_{lp}(\xi_\rho, \xi_\phi, \xi_3 + c\tau) \times \exp [i(k_3 + \kappa^{lp})x_3 - i c(k_4 - \kappa^{lp})t] \quad (25)$$

equivalent to eq. (12) where  $\xi_\rho = \sqrt{\xi_1^2 + \xi_2^2}$  and  $\xi_\phi = \text{atan2}(\xi_2, \xi_1)$ . This gives

$$\Phi_{lp} = \frac{C_{lp}^{LG} w_0}{w} \left( \frac{\sqrt{2}\xi_\rho}{w} \right)^{|l|} L_p^{|l|} \left( \frac{2\xi_\rho^2}{w^2} \right) \times \exp \left[ \frac{i2b\xi_\rho^2}{w_0^2(\xi_3 + c\tau - i2b)} + i l \xi_\phi - i g_{lp} \right], \quad (26)$$

where  $L_p^{|l|}$  are the generalized Laguerre polynomials and

$$g_{lp}(\xi_3, \tau) = (1 + |l| + 2p) \arctan \left( \frac{\xi_3 + c\tau}{2b} \right), \quad (27)$$

is the Gouy phase in terms of the radial Laguerre index  $p$  and the azimuthal index  $l$  that may be positive or negative.

The operator for the axial component of canonical OAM can be expressed as

$$\hat{L}_3 = \xi_\rho \times \hat{p}_\phi = \frac{\hbar}{i} \frac{\partial}{\partial \xi_\phi} \quad (28)$$

The Laguerre-Gaussian beam functions (25) can thus be seen to give

$$\hat{L}_3 \Psi_{lp} = l \hbar \Psi_{lp}, \quad (29)$$

showing  $L_3 = l \hbar$  are the possible eigenvalues of OAM for a Laguerre-Gaussian beam.

### III. PROBABILISTIC INTERPRETATION

In this section, the correspondence between the particle current (6) for Bateman-Hillion beams and that of the Schrödinger equation for particle beams will be investigated as means of determining the probability density of finding a particle in a Bateman-Hillion beam. As a starting point it will be useful to evaluate each component of the Bateman-Hillion particle current

$$j_\mu^{mn} = \Psi_{mn}^* \hat{j}_\mu \Psi_{mn}. \quad (30)$$

This leads to

$$j_1^{mn} = \frac{4b(\xi_3 + c\tau)\xi_1}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]} \frac{\hbar}{m_0} |\Psi_{mn}|^2, \quad (31)$$

$$j_2^{mn} = \frac{4b(\xi_3 + c\tau)\xi_2}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]} \frac{\hbar}{m_0} |\Psi_{mn}|^2, \quad (32)$$

$$j_3^{mn} = \left[ k_3 + \kappa^{mn} - \frac{2b(1 + m + n)}{(\xi_3 + c\tau)^2 + 4b^2} \right] \frac{\hbar}{m_0} |\Psi_{mn}|^2 - \frac{2b(\xi_1^2 + \xi_2^2)[(\xi_3 + c\tau)^2 - 4b^2]}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]^2} \frac{\hbar}{m_0} |\Psi_{mn}|^2, \quad (33)$$

$$j_4^{mn} = \left[ k_4 - \kappa^{mn} + \frac{2b(1 + m + n)}{(\xi_3 + c\tau)^2 + 4b^2} \right] \frac{\hbar}{m_0} |\Psi_{mn}|^2 + \frac{2b(\xi_1^2 + \xi_2^2)[(\xi_3 + c\tau)^2 - 4b^2]}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]^2} \frac{\hbar}{m_0} |\Psi_{mn}|^2, \quad (34)$$

where

$$|\Psi_{mn}|^2 = \left( \frac{C_{mn}^{HG} w_0}{w} \right)^2 H_m^2 \left( \frac{\sqrt{2}\xi_1}{w} \right) H_n^2 \left( \frac{\sqrt{2}\xi_2}{w} \right) \times \exp \left[ -\frac{8b^2(\xi_1^2 + \xi_2^2)}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]} \right]. \quad (35)$$

The continuity equation for the Klein-Gordon equation (1) is

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{1}{c} \frac{\partial j_4}{\partial t} = 0. \quad (36)$$

Eqs. (33) and (34) enable this expression to be rewritten in the form

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{1}{m_0} \left( k_3 \frac{\partial}{\partial x_3} + k_4 \frac{\partial}{\partial t} \right) |\Psi_{mn}|^2 = 0, \quad (37)$$

or equivalently

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{1}{m_0} (k_3 + k_4) \frac{\partial}{\partial t} |\Psi_{mn}|^2 = 0, \quad (38)$$

having used eq. (19). This result reduces to the simplified expression

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial}{\partial t} |\Psi_{mn}^S|^2 = 0, \quad (39)$$

in the non-relativistic limit where  $k_3 \ll k_4$  and  $m_0 c^2 \simeq \hbar k_4$ .

In an earlier paper [14] it was shown that eqs. (1) and (3) reduce to the Schrödinger equation

$$\frac{\partial^2 \Psi_{mn}^S}{\partial x_1^2} + \frac{\partial^2 \Psi_{mn}^S}{\partial x_2^2} + \frac{\partial^2 \Psi_{mn}^S}{\partial x_3^2} + 2i \frac{m}{\hbar} \frac{\partial \Psi_{mn}^S}{\partial t} = 0, \quad (40)$$

and the non-relativistic form of the Bateman-Hillion ansatz

$$\Psi_{mn}^{OS} = \Phi_{mn}^S(\xi_1, \xi_2, \tau) \exp \left[ \frac{i}{\hbar} (P_3 x_3 - E_s t) \right], \quad (41)$$

where  $E_s$  is the non-relativistic energy of the particle and

$$\Phi_{mn}^S = \int \Phi_{mn} \delta(\xi_3 - v\tau) d\xi_3. \quad (42)$$

For comparison to results in the present context  $\Psi_{mn}^{OS}$  must be further subject to the unitary transformation (8) that simplifies to

$$\Psi_{mn}^S = \Psi_{mn}^{OS} \exp \left( \frac{i N^{mn} \hbar t}{m_0 w_0^2} \right) \quad (43)$$

in the non-relativistic limit  $c \rightarrow \infty$ .

It is readily shown that eq. (39) is the continuity equation for the Schrödinger equation (40) since

$$\frac{\partial j_3}{\partial x_3} = \frac{P_s}{\hbar} \frac{\partial}{\partial x_3} |\Psi_{mn}^S|^2 = 0. \quad (44)$$

It is thus concluded from a direct comparison of eqs. (38) and (39) that

$$P_{BH} = m_0 \frac{j_3 + j_4}{k_3 + k_4} = |\Psi_{mn}|^2 \quad (45)$$

is the relativistic probability density for finding a particle in a Bateman-Hillion beam. This differs from the widely cited [27] Klein-Gordon probability density

$$P_{KG} = \frac{j_4}{c} \quad (46)$$

due to the fact  $\Psi_{mn}$  is further constrained under the parabolic equation (15). It is also of interest to notice that  $P_{BH}$  is form invariant under Lorentz transformations whereas  $P_{KG}$  is not as an isolated component of a 4-vector.

Bateman-Hillion functions can be normalized using the integral expression

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\Psi|^2 \delta(\xi_3 - v_3 \tau) d\xi_1 d\xi_2 d\tau = \frac{1}{L}, \quad (47)$$

having set the probability of finding the particle in a beam of length  $L$  is 1. This evaluates to

$$C_{mn}^{HG} = \sqrt{\frac{2}{\pi w_0^2 L 2^{m+n} m! n!}} \quad (48)$$

for Hermite-Gaussian beams; and

$$C_{lp}^{LG} = \sqrt{\frac{4p!}{w_0^2 L (p + |l|)!}} \quad (49)$$

for Laguerre-Gaussian beams.

Expectation values for the measurable properties of each particle in the beam can be calculated as

$$\langle \Psi | \hat{O} | \Psi \rangle_P = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\Psi^* \hat{O} \Psi) \delta(\xi_3 - v_3 \tau) d\xi_1 d\xi_2 d\tau, \quad (50)$$

where  $\hat{O}$  is the quantum mechanical operator for each observable quantity. Here, the subscript  $P$  has been included as a reminder that the integration is performed over a planar cross-section perpendicular to the axis of the beam but not along the axis itself.

#### IV. CALCULATION OF 4-MOMENTUM

The canonical 4-momentum operator  $\hat{p}_\mu$  is defined in eq. (2) in terms of the 4-position vector  $x_\mu$ . We next seek to use the fact  $\Psi_{mn}$  depends on  $X_\mu$  as well as  $x_\mu$  to define a distinct kinetic 4-momentum operator  $\hat{P}_\mu$  to satisfy the eigenvalue equation

$$\hat{P}_\mu \Psi_{mn} = \hbar k_\mu \Psi_{mn} \quad (51)$$

The first step is to write

$$\Phi_{mn}(\xi_1, \xi_2, \xi_3 + c\tau) = \Phi_{mn}(x_1 - X_1, x_2 - X_2, x_3 - X_3 + ct - cT) \quad (52)$$



having used eq. (4). This indicates

$$\frac{\partial \Phi_{mn}}{\partial x_\mu} = -\frac{\partial \Phi_{mn}}{\partial X_\mu} \quad (53)$$

and therefore

$$i\hbar \left( \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial X^\mu} \right) \Psi_{mn} = \hbar (k_\mu + \kappa_\mu^{mn}) \Psi_{mn} \quad (54)$$

From comparison of this expression to eq. (51) it can be seen that

$$\hat{P}_\mu \Psi_{mn} = \left( i\hbar \frac{\partial}{\partial x^\mu} + i\hbar \frac{\partial}{\partial X^\mu} - \hbar \kappa_\mu^{mn} \right) \Psi_{mn} = \hbar k_\mu \Psi_{mn} \quad (55)$$

or equivalently

$$\hat{P}_1 \Psi_{mn} = \hat{P}_2 \Psi_{mn} = 0, \quad \hat{P}_3 \Psi_{mn} = \hbar k_3 \Psi_{mn}, \quad (56)$$

$$\hat{P}_4 \Psi_{mn} = \sqrt{\hbar^2 k_3^2 + 2\kappa^{mn} (k_3 + k_4) + m_0^2 c^2} \Psi_{mn}, \quad (57)$$

having used eq. (16). These results are the eigenvalue equations for the kinetic 4-momentum of each particle in a relativistic Hermite-Gaussian beam. In completing this argument, it is necessary to find the explicit form of  $N^{mn}$  from eq. (14).

Inserting the Bateman-Hillion ansatz (12) into eq. (14) gives

$$\langle \Psi_{mn} | m_0 \hat{j}_\mu | \Psi_{mn} \rangle_P = \hbar (k_\mu + \kappa_\mu^{mn}) + \langle \Phi_{mn} | m_0 \hat{j}_\mu | \Phi_{mn} \rangle_P. \quad (58)$$

Here, the term  $\langle \Phi_{mn} | m_0 \hat{j}_\mu | \Phi_{mn} \rangle_P$  can be evaluated using the integrals

$$\int_{-\infty}^{+\infty} x H_m^2(\sqrt{\alpha}x) e^{-\alpha x^2} dx = 0, \quad (59)$$

$$\int_{-\infty}^{+\infty} x^2 H_m^2(\sqrt{\alpha}x) e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha^3}} \left( \frac{1}{2} + m \right). \quad (60)$$

The result is

$$\langle \Phi_{mn} | m_0 \hat{j}_\mu | \Phi_{mn} \rangle_P = -\hbar \kappa_\mu^{mn} \quad (61)$$

having set

$$N^{mn} = 1 + m + n. \quad (62)$$

Putting eq. (61) into (58) gives

$$\langle \Psi_{mn} | m_0 \hat{j}_\mu | \Psi_{mn} \rangle_P = \hbar k_\mu \quad (63)$$

It is thus established that the eigenvalues of the kinetic 4-momentum operator  $\hat{P}_\mu$  are equal to the expectations values for the mass current for all Hermite-Gaussian beam modes.

Equations (57) and (62) enable the total energy  $E_{HG}^{mn}$  for each particle in a Hermite-Gaussian mode to be written as

$$E_{HG}^{mn} = c \sqrt{\hbar^2 k_3^2 + \frac{2\hbar^2}{w_0^2} (1 + m + n) + m_0^2 c^2}. \quad (64)$$

Comparing this result to the energy of a free particle

$$E_{FP} = c \sqrt{\hbar^2 k_3^2 + m_0^2 c^2} \quad (65)$$

of identical mass  $m_0$  and axial wave number  $k_3$  shows that the beam particle picks up an additional energy contribution

$$\hbar^2 K_T^{mn} = \frac{2\hbar^2}{w_0^2} (1 + m + n) \quad (66)$$

where  $K_T^{mn}$  is defined in eq. (17), as a result of being localized. The remaining task is therefore to assign a physical interpretation to this term.

It can be inferred from inspection of eq. (6) that the expectation values of canonical 4-momentum and mass current must be related through the expression

$$\langle \Psi_{mn} | m_0 \hat{j}_\mu | \Psi_{mn} \rangle_P = \Re \langle \Psi_{mn} | \hat{p}_\mu | \Psi_{mn} \rangle_P \quad (67)$$

where the operator  $\Re$  takes the real part of the argument. Equations (20), (58), (61) and (67) can therefore be used together to give

$$\Re \langle \Psi_{mn} | \hat{p}_1^2 + \hat{p}_2^2 | \Psi_{mn} \rangle_P = \frac{2\hbar^2}{w_0^2} (1 + m + n) \quad (68)$$

This shows that the middle term under the square root sign in eq. (64) represents the contribution of the fluctuating transverse components of momentum to the total energy of each particle.

## V. QUANTUM POTENTIAL

The concept of distinguishing between canonical and kinetic 4-momentum has familiarity from the description [27] of a particle of charge  $e$  moving in an electromagnetic 4-potential  $A_\mu$ . The kinetic 4-momentum for this problem is

$$\hat{\pi}_\mu = \hat{p}_\mu - e A_\mu. \quad (69)$$

For the purposes of comparison the relationship between the kinetic and the canonical 4-momentum of a beam particle given in eq. (55) can be written as

$$\hat{P}_\mu = \hat{p}_\mu - m_0 \hat{U}_\mu, \quad (70)$$

where

$$\hat{U}_\mu = \frac{\hbar}{m_0} \left( \frac{1}{i} \frac{\partial}{\partial X^\mu} + \kappa_\mu^{mn} \right). \quad (71)$$

Eqs. (69) and (70) are similar in form but  $A_\mu$  is an external 4-potential whereas  $\hat{U}_\mu$  is an operator. The understanding here is that wave equations are constructed using kinetic 4-momentum to take account of external potentials and canonical 4-momentum if no external potential is present. The Hermite-Gaussian function  $\Psi_{mn}$  was derived from a wave equation that contains only canonical 4-momentum operators but it is still possible to identify a 4-potential like term  $\hat{U}_\mu$  in the definition of the kinetic 4-momentum  $P_\mu$  analogous to the role of the external 4-potential  $A_\mu$  in  $\pi_\mu$ . Equation (71) will be referred to as the 4-potential operator.

Kinetic 4-momentum was defined in eq. (51) to be a real quantity. It follows from eq. (70) that the particle current can be written in the form

$$j_\mu = \left( \frac{\hbar k_\mu}{m_0} + U_\mu \right) |\Psi|^2 \quad (72)$$

where

$$U_\mu = \frac{\hbar}{m_0} \Re \left( \frac{i}{\Psi} \frac{\partial \Psi}{\partial X^\mu} + \kappa_\mu^{mn} \right) \quad (73)$$

is a real quantum 4-potential field. Comparing eq. (72) to the component eqs. (31) through (34) gives

$$U_1^{mn} = \frac{\hbar}{m_0} \frac{4b(\xi_3 + c\tau)\xi_1}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]}, \quad (74)$$

$$U_2^{mn} = \frac{\hbar}{m_0} \frac{4b(\xi_3 + c\tau)\xi_2}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]}, \quad (75)$$

$$U_3^{mn} = \frac{\hbar}{m_0} \left[ \kappa^{mn} - \frac{2b(1+m+n)}{(\xi_3 + c\tau)^2 + 4b^2} \right] - \frac{\hbar}{m_0} \frac{2b(\xi_1^2 + \xi_2^2)[(\xi_3 + c\tau)^2 - 4b^2]}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]^2}, \quad (76)$$

$$U_4^{mn} = \frac{\hbar}{m_0} \left[ -\kappa^{mn} + \frac{2b(1+m+n)}{(\xi_3 + c\tau)^2 + 4b^2} \right] + \frac{\hbar}{m_0} \frac{2b(\xi_1^2 + \xi_2^2)[(\xi_3 + c\tau)^2 - 4b^2]}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]^2}, \quad (77)$$

to be the explicit form of the quantum 4-potential for a Hermite-Gaussian beam.

Expression (72) is a quantum mechanical equation describing a particle in a localized state. In the absence of localization ( $w_0 \rightarrow \infty$ ) it reduces to the form

$$j_\mu = \frac{\hbar k_\mu}{m_0} |\Psi|^2 \quad (78)$$

showing that the quantum 4-potential term has vanished. Squaring eq. (78) gives

$$j_M^2 - m_0^2 c^2 |\Psi|^2 = 0 \quad (79)$$

where  $j_M^2 = m_0^2 j_\mu j^\mu$ . It is clear that if we now add back the quantum 4-potential into both eqs. (78) and (79) then eq. (79) must pick up an additional scalar term  $V^2$  such that

$$j_M^2 - m_0^2 c^2 |\Psi|^2 + V^2 = 0 \quad (80)$$

where

$$V^2 = -|\hbar k_\mu - m_0 U_\mu|^2 - m_0^2 c^2 |\Psi|^2 \quad (81)$$

Expanding this expression gives

$$V^2 = -m_0^2 |U_\mu^{mn}|^2 - 2\hbar k^\mu m_0 U_\mu^{mn} - \hbar^2 K_T^{mn} = 4\hbar^2 \left[ \frac{1+m+n}{w^2} - \frac{\xi_1^2 + \xi_2^2}{w^4} \right] \quad (82)$$

having used

$$|U_\mu^{mn}|^2 = -\frac{\hbar^2}{m_0^2} \frac{16b^2(\xi_3 + c\tau)^2(\xi_1^2 + \xi_2^2)}{w_0^4[(\xi_3 + c\tau)^2 + 4b^2]^2}, \quad (83)$$

$$k^\mu U_\mu^{mn} = \frac{\hbar}{m_0} \left[ -\frac{K_T^{mn}}{2} + \frac{8b^2(1+m+n)}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]} \right] + \frac{\hbar}{m_0} \frac{8b^2(\xi_1^2 + \xi_2^2)[(\xi_3 + c\tau)^2 - 4b^2]}{w_0^4[(\xi_3 + c\tau)^2 + 4b^2]^2}, \quad (84)$$

alongside eq. (16). It is concluded from this argument that  $V^2$  is itself a quantum potential appearing in eq. (81) as the scalar analog of the quantum 4-potential  $U_\mu$  in eq. (72).

The OAM operator (28) can be rewritten in Cartesian coordinates to give

$$\hat{L}_3 = \xi_1 \hat{p}_2 - \xi_2 \hat{p}_1 \quad (85)$$

or equivalently

$$\hat{L}_3 = \xi_1 (\hat{P}_2 + m_0 \hat{U}_2) - \xi_2 (\hat{P}_1 + m_0 \hat{U}_1) \quad (86)$$

having used eq. (70). This last result simplifies to

$$\hat{L}_3 = \xi_1 m_0 \hat{U}_2 - \xi_2 m_0 \hat{U}_1 \quad (87)$$

since  $P_\mu = (0, 0, \hbar k_3, \hbar k_4)$ . It is therefore concluded that the quantum 4-potential operator  $\hat{U}_\mu$  and not the kinetic 4-momentum operator  $\hat{P}_\mu$  is the source of the mass flow resulting in OAM.

Calculating the expectation value of each component  $\hat{U}_\mu$  of the quantum 4-potential and the scalar analog  $V^2$  we obtain

$$\langle \Psi_{mn} | U_\mu | \Psi_{mn} \rangle_P = \langle \Psi_{mn} | V_\mu^2 | \Psi_{mn} \rangle_P = 0, \quad (88)$$

This result shows that quantum 4-potential is a fluctuating phenomenon. Specifically, the presence of quantum 4-potential can cause the canonical 4-momentum of a localized particle in a beam to instantaneously deviate from

the kinetic 4-momentum but it has no affect at all on the expected 4-momentum of the particle.

The original concept of a quantum potential was introduced by David Bohm [6] who started from an ansatz to solve the Schrödinger equation. This takes the form

$$\Psi = R \exp \left( i \frac{S}{\hbar} \right), \quad (89)$$

where the amplitude  $R$  and  $S/\hbar$  are real valued functions.

On inserting eq. (89) into the Schrödinger equation (40), the imaginary part of the equation can be identified as the continuity equation (39) and the real part as the Hamilton-Jacobi equation

$$-\frac{\partial S_{mn}}{\partial t} = \frac{|\nabla S_{mn}|^2}{2m_0} + Q. \quad (90)$$

where

$$Q = -\frac{\hbar^2}{2m_0} \frac{\nabla^2 R_{mn}}{R_{mn}}, \quad (91)$$

is the Bohm potential. It is of interest next to investigate how the quantum 4-potential and the Bohm potential are related to each other.

The solution to the Schrödinger equation for Hermite-Gaussian beams is given in eqs. (41) and (42). On comparing eq. (41) and (89) the explicit form of the amplitude  $R_{mn}$  and phase function  $S_{mn}$  can be read off to be

$$R_{mn} = \frac{C_{mn}^{HG} w_0}{w_S} H_m \left( \frac{\sqrt{2}\xi_1}{w_S} \right) H_n \left( \frac{\sqrt{2}\xi_2}{w_S} \right) \times \exp \left( -\frac{\xi_1^2 + \xi_2^2}{w_S^2} \right), \quad (92)$$

and

$$S_{mn} = P_3 x_3 - Et - (1 + m + n)\hbar\omega_0 t + \frac{2(\xi_1^2 + \xi_2^2)\hbar\omega_0\tau}{w_S^2} - \hbar(1 + m + n) \arctan(2\omega_0\tau), \quad (93)$$

where

$$w_S = w_0 \sqrt{1 + 4\omega_0\tau^2}, \quad \omega_0 = \frac{\hbar}{m_0 w_0^2}. \quad (94)$$

Inserting eq. (92) into eq. (91) shows the Bohm quantum potential for a non-relativistic Hermite-Gaussian beam to be

$$Q = \frac{2\hbar^2}{m_0} \left[ \frac{1 + m + n}{w_S^2} + \frac{\xi_1^2 + \xi_2^2}{w_S^4} \right]. \quad (95)$$

Eq. (95) can in turn be inserted into the Hamilton-Jacobi equation (90) giving eq. (93) as a solution.

It is clear from eqs. (82) and (95) that

$$Q = \lim_{c \rightarrow \infty} \frac{V^2}{2m_0}. \quad (96)$$

This result shows that the Bohm potential for a Hermite-Gaussian beam is the non-relativistic limit of the scalar form  $V^2$  of the relativistic quantum potential defined in eq. (82).

## VI. SUMMARY

A relativistic solution for Hermite-Gaussian particle beams presented in an earlier paper [14] has been used to calculate the properties of the particles in the beam. In the original paper, the solutions were obtained using a Bateman-Hillion ansatz that reduces the Klein-Gordon equation to a parabolic form thus enabling  $|\Psi|^2$  to be interpreted as the probability density for finding the particle. It was shown the solutions are form preserving under Lorentz transformations and correspond to those of the Schrödinger equation in the non-relativistic limit. It was also shown the solutions take account of the Gouy phase in the beam.

In this paper, a Lorentz covariant kinetic 4-momentum operator has been introduced equal to canonical 4-momentum operator minus a quantum 4-potential term. The quantum 4-potential originates at the beam waist where it introduces fluctuating terms into the canonical 4-momentum of transversely localized particles. All the eigenvalues of the kinetic 4-momentum operator have in fact been shown to equal the expectation values of the real parts of the canonical 4-momentum components. The total energy of a particle for each beam mode has also been calculated. It has been found, in particular, that the energy of a particle in a beam differs from the energy of a free particle as a result of fluctuating transverse momentum components in the spatial plane perpendicular to the axis of the beam.

Transverse momentum is needed to explain both the divergence of the beam after passing through the beam waist as well as OAM. Here, solutions have been presented for Laguerre-Gaussian modes to demonstrate the possibility for OAM in the Bateman-Hillion formalism. It has also been found that in our proposed partitioning of canonical 4-momentum into kinetic and quantum 4-potential parts that the kinetic part makes no contribution to OAM meaning that OAM is a pure manifestation of quantum 4-potential. A clear indicator to this is that particles must be localized to exhibit OAM. Free particles cannot have OAM in the absence of localization since they have no quantum 4-potential.

The quantum 4-potential has been discussed in relation to the electromagnetic 4-potential and the Bohm potential. Quantum 4-potential acts on mass in an analogous manner to how electromagnetic 4-potential operates on



charge. Specifically, both potentials operate to produce a distinction between the canonical 4-momentum of a particle that includes the influence of the 4-potential and a kinetic 4-momentum that does not. It is clear though that the quantum and electromagnetic 4-potentials are, at least, different in the sense that a charged particle can intrinsically generate an electromagnetic field whereas it is the localization of a particle that indicates quantum 4-potential and not just the presence of the particle by itself.

The Bohm potential and quantum 4-potential related concepts that both vanish in the absence of localization and have null expectation values. The quantum 4-potential has been developed here in the context of relativistic Hamiltonian quantum mechanics. By contrast the Bohm potential was first identified as a term in a quantum form of the non-relativistic Hamilton-Jacobi equation that Bohm derived from the real part of the Schrödinger equation. It has been demonstrated here that if both sides of a quantum mechanical equation containing quantum 4-potential are squared then the quantum potential in the derived equation is a scalar term. The Bohm potential is simply the non-relativistic limit of this scalar counterpart of quantum 4-potential.

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